**Indian Statistical Institute**
**M.Math Second year**
**Final exam - Partial differential Equations**

Total Marks : 40

1. (a) Define the normed linear space $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ for $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. [3]
   (b) Does there exist a function $f : I\!R \to I\!R$ which is in $H^1(I\!R \setminus \{0\})$ but not in $H^1(I\!R)$. [2]

2. (a) State Rellich-Kondrachov compactness theorem for $W^{1,p}(\Omega)$ where $\Omega$ is a bounded open set of $I\!R^N$ with $C^1$ boundary and $1 \leq p < n$. Deduce that $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$ for all $1 \leq p \leq \infty$. [3]
   (b) State and prove Poincare's inequality for $W_0^{1,2}(\Omega)$. [4]

3. (a) Let $\Omega \subset I\!R^N$ be a bounded open set, $f \in L^2(\Omega)$ and $u \in H^1(\Omega) \cap C(\overline{\Omega})$ such that
$$\int_\Omega \nabla u.\nabla v + \int_\Omega uv = \int_\Omega fv$$
   for every $v \in H_0^1(\Omega)$. Show that
$$\min\{\inf_\Gamma u, \inf_\Omega f\} \leq u(x) \leq \max\{\sup_\Gamma u, \sup_\Omega f\} \text{ for every } x \in \Omega.$$
   Here $\Gamma$ denotes the boundary of $\Omega$. [5]
   (b) If $f = 0$, show that $\|u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Gamma)}$. [1]

4. Let $\Omega_1$ and $\Omega_2$ be bounded domains in $I\!R^N$ such that $\Omega_1 \subset \Omega_2$. Let $\lambda_1(\Omega_i)$ denote the least eigenvalue of Laplace operator with Dirichlet boundary conditions in $\Omega_i$. Then show that $\lambda_1(\Omega_1) \geq \lambda_2(\Omega_2)$. [5]

5. (a) State the trace for $W^{1,p}(\Omega)$. Clearly state the assumptions on the domain $\Omega$. [3]
   (b) Let $B_1(0)$ denote the unit ball in $R^N$ centered at 0 and $\Omega = B_1(0) \cap R_+^N$. If $u(x) = |x|^{-\alpha}$ for $0 < \alpha < \frac{N-2}{2}$, then it is known that $u \in W^{1,2}(\Omega)$. Find the trace of $u$ and explain. [4]

6. (a) Define the fundamental solution of Heat equation in $I\!R^N$. [2]
   (b) State Duhamel's principle for heat equation. [3]

7. Assume $\Omega$ is connected. A function $u \in H^1(\Omega)$ is a weak solution of Neumann's problem
$$(*) \begin{cases} -\Delta u &= f \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial\Omega \end{cases}$$
   if
$$\int_\Omega \nabla u.\nabla v dx = \int_\Omega fv dx$$
   for all $v \in H^1(\Omega)$. Suppose $f \in L^2(\Omega)$, prove that (*) has a weak solution if and only if $\int_\Omega f dx = 0$. [5]